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Minimizing the entropy production in heat exchange

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Abstract

We present a theoretical proof that the entropy production due to heat exchange in a heat exchanger is minimum when the local entropy production is constant in all parts of the system. The solution for the minimum is independent of the value of the heat transfer coefficient. The general case is compared to the minimization problem that has equipartition of forces as solution. It is found that equipartition of forces predicts the minimum from the general solution well for typical heat exchange conditions. The discrepancy between the two solutions depends largely on the temperature dependency of the heat transfer coefficient. The optimal heat exchange conditions are very well approximated in practice with a counter-current heat exchanger; since the minimum in the entropy production space probably is flat. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

Bejan [1] and coworkers have studied the minimization of entropy production (or generation) in many processes that include heat and mass transfer (see [1,2] for further references). We are interested in the nature of the solution to such minimization problems [3]. Are there characteristic features of the state that has minimum entropy production, and if so, what are they? In a recent study [4], we reported that one path that gave minimum entropy production in a heat exchanger was characterized by equipartition of the inverse temperature difference, $\Delta(1/T)$. The inverse temperature difference is the thermodynamic force for heat transport. Knowlegde about the nature of the driving forces in the state of minimum entropy production may be useful for design of new apparatuses. A question about the general nature of the constant force solution is therefore appropriate.

In this work we shall find a more general solution to the minimization problem, and discuss how and why the equipartition of forces solution deviates from the general solution. We shall see that the mathematical assumption that was made in the derivation of equipartition of forces (that the functional derivative could be replaced by a local derivative) has a physical translation: The constant force solution can only be expected if the system lacks memory of prehistory or a function that connects the different parts of the system to each other. If this is not so, the state of minimum entropy production is characterized by constant entropy production.

A solution giving equipartition of entropy production was presented by Tondeur and co-workers [5,6]. They assumed a constant heat transfer coefficient, so their result will also give equipartition of forces. In this work we let the heat transfer coefficient vary with the temperature, and include as memory function, the energy balance. We ask, given that the energy balance must be fulfilled, how can we specify boundary conditions that are compatible with minimum entropy production in the total system? And, how can the state of the system be characterised in this case?

2. The system

The system is the same as presented before [4], see Fig. 1. It consists of a hot and a cold fluid separated by a thin metal plate. Both fluids are perfectly mixed in the y-direction. We shall therefore discuss changes per unit length in the y-direction, $\Delta y = 1$. In the x-direction,

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Nomenclature

C	constant (J/K m)
C_p	constant pressure heat capacity (J/K kg)
EoF	equipartition of forces
EoEP	equipartition of entropy production rate
J	fluid flow (kg/s)
J_i	flux
J'_q	measurable heat flux (J/m^2 s)
L	integrand in the Euler–Lagrange equation
l	length of the unit (m)
$\bar{l}_{qq}(T_\mathrm{h})$	average heat transfer coefficient (J K/m^2 s)
q	fraction of total heat transferred (J/s)
Q	total heat transferred (J/s)
$R(T_\mathrm{h})$	thermal resistance (m^2 s/J K)
$\mathrm{d}S_\mathrm{irr}/\mathrm{d}t$	total entropy production rate (J/K s)
$\Delta(1/T)$	thermal force (1/K)
T	temperature (K)
T_c	temperature of the cold fluid (K)
T_h	temperature of the hot fluid (K)
$T_\mathrm{h,in}$	inlet temperature of the hot fluid (K)
$T_\mathrm{h,out}$	outlet temperature of the hot fluid (K)
x, y, and z	space coordinates (m)
$X = \Delta(1/T)$	thermal force (1/K)
X_i	force conjugate to J_i
Δ	difference
λ	Lagrange multiplier
σ	local entropy production rate (J/K m^3 s)

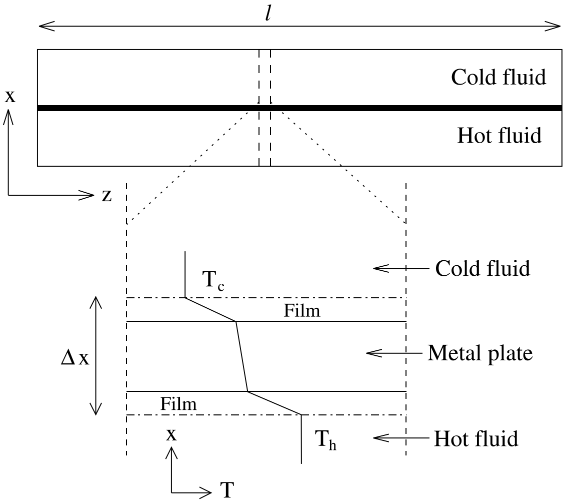

Fig. 1. The heat exchange example showing coordinate axes. The hot fluid flows from left to right. A cold fluid flows on the other side (direction not specified). There is a temperature difference between the bulk phases of the two fluids, $T_\mathrm{h} - T_\mathrm{c}$.

there is a temperature gradient in the metal plate and in the fluid films next to the plate as shown in the lower part of Fig. 1. In the bulk of both fluids there are only gradients in the z-direction. Heat is conducted through the metal plate in the x-direction. The conduction of heat in other directions is neglected. The system has length l, so $0 < z < l$.

The hot fluid flows from left to right. It enters at temperature $T_\mathrm{h,in}$ and leaves at $T_\mathrm{h,out}$. The mass flow rate, J, and the heat capacity, C_p, are known. The temperature profile of the hot fluid, $T_\mathrm{h}(z)$, is then given by conservation of energy

$$JC_\mathrm{p}\,\mathrm{d}T_\mathrm{h} = J'_q \Delta y\,\mathrm{d}z, \tag{1}$$

where J'_q is the z-dependent heat flux across the metal plate.

None of the properties of the cold fluid (fluid flow, flow direction, heat capacity, etc.) are specified yet. We assume that we are able to control the temperature of the cold fluid at every position, z, in some way. We will discuss later how to achieve this control in practice.

3. The minimization problem

The aim is to minimize the total entropy production rate, $\mathrm{d}S_\mathrm{irr}/\mathrm{d}t$, of the system for a given amount of heat transferred, Q (equivalent to fixing $T_\mathrm{h,in}$ and $T_\mathrm{h,out}$). The local entropy production rate is the product sum of all fluxes (J_i) and their conjugate forces (X_i) in the system [7–9]

$$\sigma = \sum_i J_i X_i. \tag{2}$$

We assume that the entropy production due to heat transport can vary, while frictional losses due to fluid flow are constant. The interesting part of σ is then

$$\sigma = J'_q \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{T}\right), \tag{3}$$

where T is the absolute temperature, $\mathrm{d}/\mathrm{d}x(1/T)$ is the thermal driving force, and J'_q is the measurable heat flux in the x-direction. In our model we deal with stationary states. In the x-direction, this gives

$$\frac{\mathrm{d}}{\mathrm{d}x} J'_q(z) = 0. \tag{4}$$

This means that the local entropy production rate, Eq. (3), can be integrated across the metal plate

$$\sigma\,\Delta x = J'_q \Delta\left(\frac{1}{T}\right). \tag{5}$$

The flux–force relation is accordingly

$$J'_q = \bar{l}_{qq}(T_\mathrm{h})\Delta\left(\frac{1}{T}\right), \tag{6}$$

where T_h is the temperature of the hot fluid. A discussion on the linearity of this equation was given previously [4]. We shall use the inverted form

$$X \equiv \Delta\left(\frac{1}{T}\right) = R(T_\mathrm{h})J'_q, \tag{7}$$

where $R(T_\mathrm{h})$ is the thermal resistance. When this is introduced into Eq. (5), we get the entropy production rate across the metal as a function of the heat flux

$$\sigma = \frac{1}{\Delta x}R(T_\mathrm{h})\left[J'_q\right]^2. \tag{8}$$

The total entropy production rate is found by integrating σ over the area of heat transfer, where we are interested in the variation between $z = 0$ and $z = l$

$$\frac{\mathrm{d}S_\mathrm{irr}}{\mathrm{d}t} = \Delta x\Delta y\int_0^l \sigma\,\mathrm{d}z = \Delta y\int_0^l R(T_\mathrm{h})\left[J'_q\right]^2\mathrm{d}z. \tag{9}$$

The constant heat flux across the metal, J'_q, is a function of z via the energy balance, Eq. (1). By introducing

$$J'_q = JC_\mathrm{p}\frac{1}{\Delta y}\frac{\mathrm{d}T_\mathrm{h}}{\mathrm{d}z} \tag{10}$$

into Eq. (9), the total entropy production rate becomes

$$\frac{\mathrm{d}S_\mathrm{irr}}{\mathrm{d}t} = \int_0^l \frac{R(T_\mathrm{h})}{\Delta y}\left[JC_\mathrm{p}\frac{\mathrm{d}T_\mathrm{h}}{\mathrm{d}z}\right]^2\mathrm{d}z, \tag{11}$$

where in addition to R, also C_p may be a function of the temperature T_h but not T_c.

The system has a minimum in Eq. (11), and this can now be found from variational calculus. If a function $T_\mathrm{h}(z)$ is a local minimizer to the functional $\mathrm{d}S_\mathrm{irr}/\mathrm{d}t$, then according to variational calculus [10], the integrand must satisfy

$$L - \frac{\mathrm{d}T}{\mathrm{d}z}\frac{\partial}{\partial(\mathrm{d}T_\mathrm{h}/\mathrm{d}z)}L = C, \tag{12}$$

where L is the integrand in Eq. (11), and C is a constant. In the present case, this gives as condition for minimum total entropy production rate, that the *local value of the entropy production rate is constant* throughout the system (equipartition of entropy production or EoEP)

$$\begin{aligned}&\frac{R(T_\mathrm{h})}{\Delta y}\left[JC_\mathrm{p}\frac{\mathrm{d}T_\mathrm{h}}{\mathrm{d}z}\right]^2 - \frac{\mathrm{d}T_\mathrm{h}}{\mathrm{d}z}\left[\frac{2R(T_\mathrm{h})}{\Delta y}\left[JC_\mathrm{p}\right]^2\frac{\mathrm{d}T_\mathrm{h}}{\mathrm{d}z}\right]\\ &\quad = -\frac{R(T_\mathrm{h})}{\Delta y}\left[JC_\mathrm{p}\frac{\mathrm{d}T_\mathrm{h}}{\mathrm{d}z}\right]^2 = -\Delta x\Delta y\sigma = C.\end{aligned} \tag{13}$$

Equipartition of forces was obtained from Euler–Lagrange optimization with the restriction that a constant amount of heat was transferred

$$Q = \Delta y\int_0^l J'_q\,\mathrm{d}z. \tag{14}$$

In order to see the connection between the two optimization methods, consider again the proof that lead to the equipartition of forces solution

$$\frac{\delta}{\delta J'_q}\int\left(R\left[J'_q\right]^2 + \lambda J'_q\right)\mathrm{d}z = 0. \tag{15}$$

The condition for the solution of constant force $RJ'_q = X = -\lambda/2$ was that the functional derivative could be replaced by the local derivative, or rather, that R was no function of J'_q. This is the same assumption used by Tondeur and Kvaalen [5] in their equipartition of entropy production solution. In our case we have such a dependency. The resistance R depends on J'_q, because it depends on the amount of heat that has been exchanged up to the point z. We therefore define q as a fraction of Q

$$q = \Delta y\int_0^z J'_q\,\mathrm{d}z', \tag{16}$$

where $z \leqslant l$. The solution to Eqs. (15) and (16) was derived by Bedeaux et al. [3]

$$X = -\frac{\lambda}{2} - \frac{1}{2}\int_z^l \frac{\mathrm{d}R}{\mathrm{d}q}\Delta y\left[J'_q\right]^2\mathrm{d}z'. \tag{17}$$

The derivative of the force, as given in Eq. (17), with respect to z is

$$\frac{\mathrm{d}X}{\mathrm{d}z} = \frac{1}{2}\frac{\mathrm{d}R}{\mathrm{d}q}\Delta y\left[J'_q\right]^2 = \frac{1}{2}\frac{\mathrm{d}R}{\mathrm{d}z}J'_q. \tag{18}$$

In the transformation, they used $\mathrm{d}q/\mathrm{d}z = \Delta yJ'_q$. The derivative of the force, as given in Eq. (7), with respect to z is

$$\frac{\mathrm{d}X}{\mathrm{d}z} = \frac{\mathrm{d}R}{\mathrm{d}z}J'_q + R\frac{\mathrm{d}J'_q}{\mathrm{d}z}. \tag{19}$$

They equated Eqs. (18) and (19), and multiplied the new equation by $2J'_q$, and obtained, after some rearrangement

$$\begin{aligned}&\left[J'_q\right]^2\frac{\mathrm{d}R}{\mathrm{d}z} + 2RJ'_q\frac{\mathrm{d}J'_q}{\mathrm{d}z} = 0,\\ &\frac{\mathrm{d}}{\mathrm{d}z}\left(R\left[J'_q\right]^2\right) = 0,\\ &\frac{\mathrm{d}}{\mathrm{d}z}\sigma = 0.\end{aligned} \tag{20}$$

The local entropy production rate is constant along the system, when the total entropy production rate is minimum. The same result was found in Eq. (13). Clearly, the local condition Eq. (10) can either enter directly in the variational calculus, as in Eqs. (11)–(13), or it can be used in an Euler–Lagrange formulation of the problem, as in Eqs. (15)–(20). The two ways to the same result are equivalent. This gives the physical insight about the memory of the system, as represented by $dR/dq \neq 0$. The equipartition of forces solution [4,11] is correct when there is no inter-dependency between the fluxes, J'_q, or when $dR/dq \neq 0$. By comparing results from the EoEP and EoF solutions, we can test how good the assumption is.

4. Calculations

A model system was used to illustrate the difference between the EoF and EoEP solutions, namely the system shown in Fig. 1. The length, l, and the width, Δy, were 10 and 1 m, respectively. The hot stream had a mass flow of $J = 0.3$ kg/s, the heat capacity of the hot stream was $C_p = 4000 + T_h$ J/kg K, and $\bar{l}_{qq} = 370\,T_h^2$ J K/m^2 s was used to estimate the heat transfer coefficient. The resistance, $R(T_h)$, is the inverse of $\bar{l}_{qq}$. The outlet temperature of the hot stream, $T_{h,out}$, was kept at 400 K.

Two cases were calculated for both EoEP and EoF. Only the inlet temperature of the hot stream was varied between the two cases. In the first case, we gave it a realistic value, $T_{h,in} = 450$ K. In the second, we gave it a high and unrealistic value, $T_{h,in} = 1000$ K, in order to show the difference between the two solutions, and when the assumption behind the EoF solution produces a significant error in the results.

The EoEP solution derived in this work requires the local entropy production, σ, to be constant. To solve this numerically, we isolated dT/dz from Eq. (13)

$$\frac{dT_h}{dz} = -\frac{\sqrt{-C}}{JC_p\sqrt{R(T_h)}}. \tag{21}$$

The negative root was the only real solution. For a random value of C, this equation was integrated over l using the constant $T_{h,out} = 400$ K at $l = 10$ m as the initial value. At $l = 0$ m, the calculated value of $T_{h,in}$ was compared to the specified value (i.e., 450 or 1000 K), and the constant C was increased or decreased, using a binary search algorithm, until they were identical.

The EoF solution however, requires the inverse temperature difference, $\Delta(1/T)$, to be constant. We solved dT_h/dz from the energy balance, Eq. (1), and introduced Eq. (6) into it. When we substituted the inverse temperature difference in the resulting equation with $-\lambda/2$, we got

$$\frac{dT_h}{dz} = -\frac{\lambda}{2JC_pR(T_h)}. \tag{22}$$

The numerical approach to solving this equation was identical to the one described above, except that here we varied the Lagrange multiplier, λ, in the binary search.

For both cases, in addition to calculating the T_h profiles as described above, we calculated the total entropy production, dS_{irr}/dt, and T_c and σ profiles.

5. Results and discussion

The temperature profiles and local entropy production profiles for the first case are shown in Figs. 2 and 3. From Fig. 2, we see that the temperature profiles are close to identical, so we expect the same from the total entropy production of the two solutions. This is indeed the case. The total entropy production for the EoF solution was $(dS_{irr}/dt)_{EoF} = 7.197$ J/K s, while for the EoEP solution, it was $(dS_{irr}/dt)_{EoEP} = 7.189$ J/K s.

Fig. 3 shows the local entropy productions throughout the system for the two different solutions. It also shows that compared to the EoEP solution, the EoF solution gives a local entropy production that is too high in the first half and too low in the second half. This is also indicated by the T-profiles in Fig. 2. We see that $|\Delta(1/T)_{EoF}|$ is slightly smaller than $|\Delta(1/T)_{EoEP}|$ at $l = 0$ m, and slightly larger at $l = 10$ m.

The temperature profiles and local entropy production profiles for the second case are shown in Figs. 4 and 5. In this case, the results are different. Looking at Fig. 4, we see a larger discrepancy between the temperature profiles. The profiles from the EoF solution are more curved than the ones found using the EoEP solution. The total entropy production was $(dS_{irr}/dt)_{EoF} = 516.5$ J/K s for the EoF solution and $(dS_{irr}/dt)_{EoEP} = 481.5$ J/K s for the EoEP solution. EoF fails to predict the minimum by 7%. A deviation was expected, since an increased temperature difference over the hot stream makes the resistance, $R(T_h)$, vary much more. Table 1

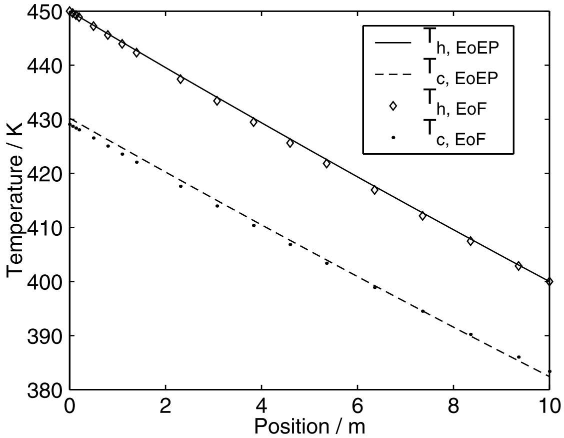

Fig. 2. Temperature profiles corresponding to the solutions given by the equipartition of entropy production (EoEP) solution (lines) and to the equipartition of force (EoF) solution (points) at $T_{h,in} = 450$ K.

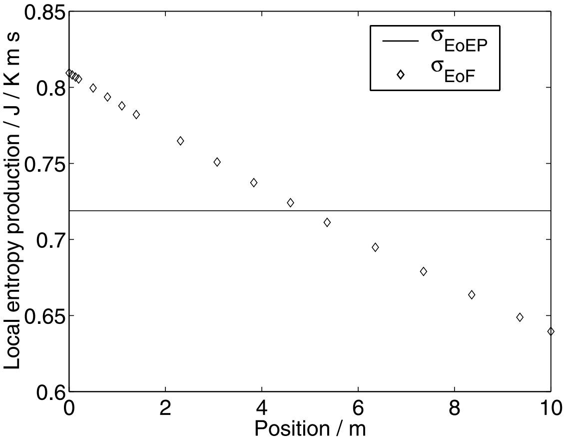

Fig. 3. The local entropy production rate as a function of position along the system at $T_{\mathrm{h,in}} = 450$ K.

shows the maximum and minimum values of R in the two cases.

In the first case, R varies by about 20%, while in case two, its variation is close to an order of magnitude. Thus, for case two, the assumptions behind the EoF solution are not adequate. However, it is remarkable that the two solutions give so similar results for the total entropy production in spite of the large variations in R and the large discrepancies in the temperature profiles from the two solutions. This can be explained by a possible flat minimum in $\mathrm{d}S_{\mathrm{irr}}/\mathrm{d}t$.

The local entropy production in the EoF solution is not constant throughout the system; see Figs. 3 and 5. If we compare the two figures, we see that local entropy production profile goes from being almost linear to being quite curved when the temperature difference over the hot stream increases from 50 to 600 K. The local entropy production profiles show that the nature of the two solutions differs more than the corresponding values for the total entropy productions suggest. On the left- and right-hand sides of the figures, respectively, we see that the profiles from the EoF solution have a positive and a negative deviation from the profiles predicted by the EoEP solution. These areas almost cancel each other, and make the entropy productions similar. This canceling and similar results obtained by others [12] supports the hypothesis of a flat minimum in the total entropy production space.

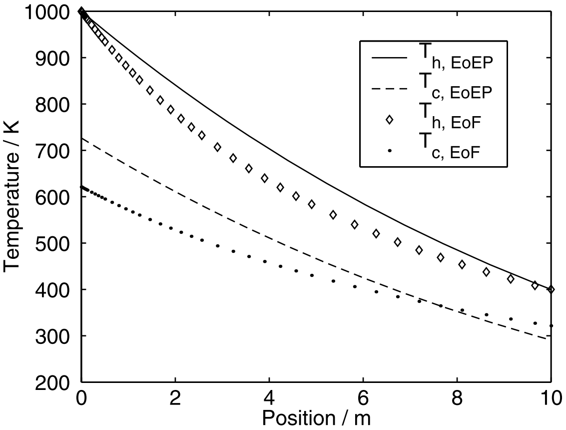

Fig. 4. Temperature profiles corresponding to the solutions given by equipartition of entropy production (EoEP) (lines) and to equipartition of force (EoF) (points) at $T_{\mathrm{h,in}} = 1000$ K.

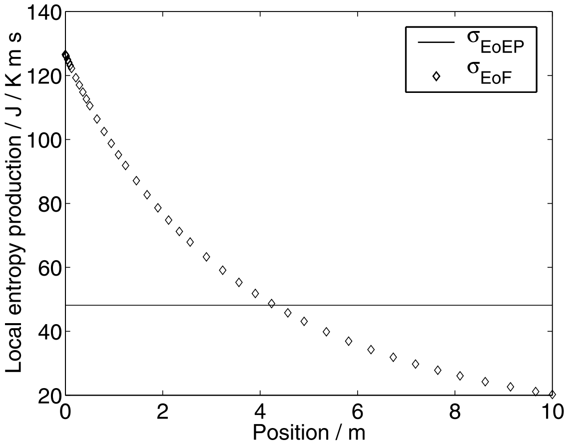

Fig. 5. The local entropy production rate as a function of position along the system at $T_{\mathrm{h,in}} = 1000$ K.

Table 1
The maximum and minimum values of R in the two calculated cases

	R_{min} (m^2 s/J K)	R_{max} (m^2 s/J K)
Case 1	1.45×10^{-6}	1.84×10^{-6}
Case 2	0.294×10^{-6}	1.84×10^{-6}

The EoF solution seems to be a good approximation to the EoEP solution for most practical purposes. From the calculations presented in this work, it seems that we may expect a deviation of less than one percent for practical cases.

The very important question now is how to achieve optimal heat exchange conditions in practice, for instance, in a heat exchanger. Our results tell that the cold fluid must have a tailored heat capacity. We must therefore think design of heat exchanger in terms of ways to obtain the target heat capacity. One possibility to achieve the target heat capacity is to use a counter-current heat exchanger with multiple inlets and outlets for the cold fluid. Another option is to use pure cross-current flow with infinitely many cold streams having different temperatures. Both alternatives are probably impractical and not economically feasible. We therefore have find a practical approximation to the optimal heat exchange solution. The standard counter-current heat exchanger is the best first approximation. It has qualitatively the same properties as the optimal solutions presented here; when the temperature difference between the hot and the cold fluids is approximately constant. The fact that the entropy production minimum of heat exchange probably is flat, suggests that there is little reason to try to approach the optimal heat exchange solution more closely in practice.

6. Conclusion

We have shown that the temperature profiles through a system with heat exchange that give minimum entropy production for the heat exchange process, correspond to equipartition of entropy production, rather than to equipartition of thermal driving force. The theoretical result can be used as an argument that heat exchange processes characterized by constant entropy production give the best second law efficiency possible. However, in practical situations, a constant driving force seems to estimate the real solution within an error less than one percent. The optimal heat exchange conditions are therefore well approximated in practice with a countercurrent heat exchanger with an approximately constant temperature difference between the fluids. The explanation for this is that the minimum in the entropy production space probably is flat.

Acknowledgements

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